Homework Assignment 2 Answers – 600.445 Fall 2002

Question 1

Consider the optical position measurement system shown in Figure 1. The system resembles the Northern Digital Polaris™ system, among others. It has two 2D CCD cameras that can detect the image of markers in the field of view of the detectors. Two coordinate systems, $\mathbf{F}_L$ and $\mathbf{F}_R$, are associated with the two detector lenses. Image processing software in the system is able to compute the positions in image coordinates of points in the field of view of the detectors. For the purposes of this exercise, we will assume that the detectors obey a pinhole camera model:

$$\vec{\mathbf{u}} = \left[u_x, u_y \right] = \left[f\frac{b_x}{b_z}, f\frac{b_y}{b_z} \right]$$

$$\vec{\mathbf{v}} = \left[v_x, v_y \right] = \left[f\frac{c_x}{c_z}, f\frac{c_y}{c_z} \right]$$

where $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are the positions of a 3D point relative to the left and right camera lens coordinate systems, respectively. The two detector coordinate systems are nominally related to each other by the relationship

$$\mathbf{F}_R = \mathbf{F}_L \cdot \mathbf{F}_{LR}$$

$$\mathbf{F}_{LR} = \left[\mathbf{I}, [X, 0, 0]^T \right]$$

where X is the distance between the detectors. But the actual relationship is slightly different:

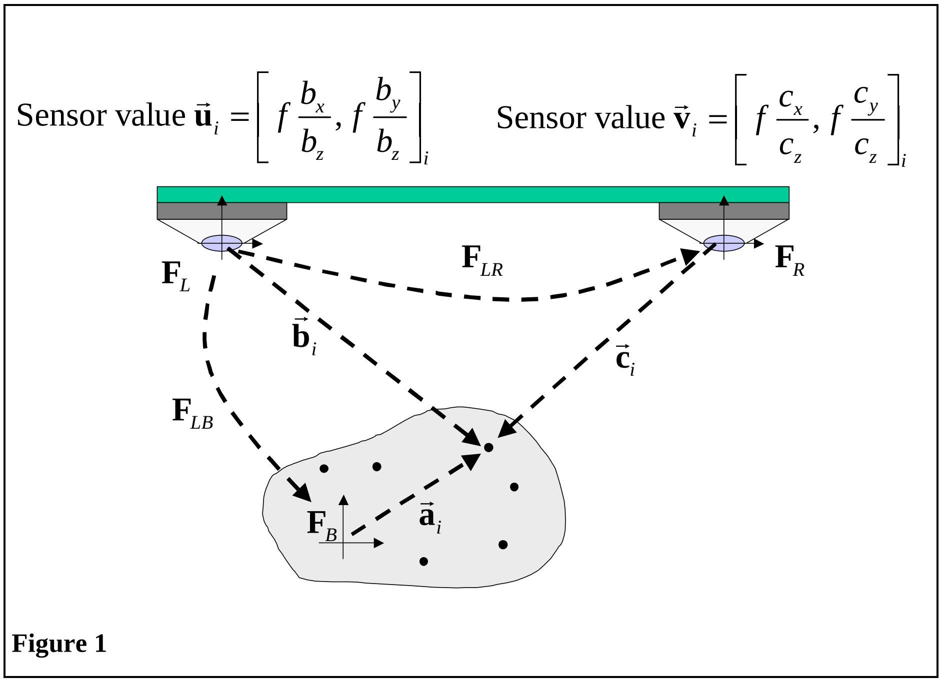

Figure 1

where $\vec{\alpha}$ and $\vec{\varepsilon}$ are small vectors

To calibrate this device, we have a calibration fixture consisting of a number N_{markers} (N>5) of non-collinear markers located at points $\vec{\mathbf{a}}_i$ with respect to the fixture coordinate system. The fixture is located at $\mathbf{F}_{LB}$ relative to $\mathbf{F}_L$.

A. (5 points) give formulas for $\vec{\mathbf{b}}_i$ and $\vec{\mathbf{c}}_i$ in terms of $\vec{\mathbf{a}}_i$, $\Delta\mathbf{R}(\vec{\alpha})$, and $\vec{\varepsilon}$.

$$\begin{aligned}
\vec{\mathbf{b}}_i &= \mathbf{F}_{LB} \cdot \vec{\mathbf{a}}_i = \mathbf{R}_{LB} \cdot \vec{\mathbf{a}}_i + \vec{\mathbf{p}}_{LB} \\
\vec{\mathbf{c}}_i &= \mathbf{F}_{\mathbf{LR}}^{-1} \cdot \vec{\mathbf{b}}_i \\
&= \mathbf{R}_{\mathbf{LR}}^{-1} \cdot \left(\vec{\mathbf{b}}_i - \vec{\mathbf{p}}_{\mathbf{LR}}\right) \text{ where } \vec{\mathbf{p}}_{\mathbf{LR}} = [X,0,0]^T \\
&= \Delta\mathbf{R}(\vec{\alpha})^{-1} \cdot \left(\vec{\mathbf{b}}_i - \vec{\mathbf{p}}_{\mathbf{LR}} - \vec{\varepsilon}\right) \\
&= \Delta\mathbf{R}(\vec{\alpha})^{-1} \cdot \left(\mathbf{F}_{LB} \cdot \vec{\mathbf{a}}_i - \vec{\mathbf{p}}_{\mathbf{LR}}\right) - \Delta\mathbf{R}(\vec{\alpha})^{-1} \cdot \vec{\varepsilon}
\end{aligned}$$

B. (10 points) Simplify these expressions so that the elements of $\vec{\alpha}$ and $\vec{\varepsilon}$ appear only linearly.

$$\begin{aligned}
\vec{\mathbf{b}}_i &= \mathbf{F}_{LB} \cdot \vec{\mathbf{a}}_i = \mathbf{R}_{LB} \cdot \vec{\mathbf{a}}_i + \vec{\mathbf{p}}_{LB} \\
\vec{\mathbf{c}}_i &= \Delta\mathbf{R}(\vec{\alpha})^{-1} \cdot \vec{\mathbf{b}}_i - \Delta\mathbf{R}(\vec{\alpha})^{-1} \cdot \vec{\mathbf{p}}_{\mathbf{LR}} - \Delta\mathbf{R}(\vec{\alpha})^{-1} \cdot \vec{\varepsilon} \\
&\approx \left(\mathbf{I} - skew(\vec{\alpha})\right) \cdot \vec{\mathbf{b}}_i - \left(\mathbf{I} - skew(\vec{\alpha})\right) \cdot \vec{\mathbf{p}}_{\mathbf{LR}} - \left(\mathbf{I} - skew(\vec{\alpha})\right) \cdot \vec{\varepsilon} \\
&= \vec{\mathbf{b}}_i - \vec{\alpha} \times \vec{\mathbf{b}}_i - \vec{\mathbf{p}}_{\mathbf{LR}} + \vec{\alpha} \times \vec{\mathbf{p}}_{\mathbf{LR}} - \vec{\varepsilon} + \vec{\alpha} \times \vec{\varepsilon} \\
&\approx \vec{\mathbf{b}}_i - \vec{\mathbf{p}}_{\mathbf{LR}} + \vec{\alpha} \times \left(\vec{\mathbf{p}}_{\mathbf{LR}} - \vec{\mathbf{b}}_i\right) - \vec{\varepsilon} \\
&= \left(\mathbf{F}_{LB} \cdot \vec{\mathbf{a}}_i - \vec{\mathbf{p}}_{\mathbf{LR}}\right) + \vec{\alpha} \times \left(\vec{\mathbf{p}}_{\mathbf{LR}} - \mathbf{F}_{LB} \cdot \vec{\mathbf{a}}_i\right) - \vec{\varepsilon} \\
&= \vec{\mathbf{d}}_i + \vec{\alpha} \times \vec{\mathbf{d}}_i - \vec{\varepsilon} \text{ where } \vec{\mathbf{d}}_i = \mathbf{F}_{LB} \cdot \vec{\mathbf{a}}_i - \vec{\mathbf{p}}_{\mathbf{LR}} \\
&= \vec{\mathbf{d}}_i - \vec{\mathbf{d}}_i \times \vec{\alpha} - \vec{\varepsilon}
\end{aligned}$$

C. (10 points) Suppose that you are given the values of $\mathbf{F}_{LB}, \vec{\mathbf{a}}_i$, $\vec{\mathbf{u}}_i$, $\vec{\mathbf{v}}_i$, and X. Outline a method for estimating $\mathbf{R}(\vec{\alpha})$ and $\vec{\varepsilon}$. Include a statement of the algorithm in some suitable pseudo-code or step-by-step summary, together with all necessary formulas.

Observe that we know the values for $\vec{\mathbf{b}}_i$ and, thus, $\vec{\mathbf{d}}_i = \vec{\mathbf{b}}_i - \vec{\mathbf{p}}_{LR}$. Expanding gives us

$$\begin{bmatrix} c_{x,i} \\ c_{y,i} \\ c_{z,i} \end{bmatrix} = \begin{bmatrix} d_{x,i} + d_z\alpha_y - d_{y,i}\alpha_z - \varepsilon_x \\ d_{y,i} + d_x\alpha_z - d_{z,i}\alpha_x - \varepsilon_y \\ d_{z,i} + d_y\alpha_x - d_{x,i}\alpha_y - \varepsilon_z \end{bmatrix}$$

Further, we know that

$$\begin{aligned}
fc_x - v_x c_z &= 0 \\
fc_y - v_y c_z &= 0
\end{aligned}$$

Substituting gives

$$\begin{aligned}
f\left(d_{x,i} + d_z\alpha_y - d_{y,i}\alpha_z - \varepsilon_x\right) - v_x\left(d_{z,i} + d_y\alpha_x - d_{x,i}\alpha_y - \varepsilon_z\right) &= 0 \\
f\left(d_{y,i} + d_x\alpha_z - d_{z,i}\alpha_x - \varepsilon_y\right) - v_y\left(d_{z,i} + d_y\alpha_x - d_{x,i}\alpha_y - \varepsilon_z\right) &= 0
\end{aligned}$$

I.e.,

$$\begin{aligned}
fd_z\alpha_y - fd_{y,i}\alpha_z - f\varepsilon_x - v_x d_{z,i} + v_x d_y\alpha_x - v_x d_{x,i}\alpha_y - v_x\varepsilon_z &= -fd_{x,i} \\
fd_x\alpha_z - fd_{z,i}\alpha_x - f\varepsilon_y - v_y d_{z,i} + v_y d_y\alpha_x - v_y d_{x,i}\alpha_y - v_y\varepsilon_z &= fd_{y,i}
\end{aligned} \qquad (*)$$

So the appropriate method might be the following:

Step 1: For each i, compute $\vec{\mathbf{d}}_i$ as shown above

Step 2: Solve the system (*) using a linear least squares solver to estimate $\vec{\alpha}, \vec{\varepsilon}$

Step 3: Compute $\Delta\mathbf{R}(\vec{\alpha})$

Step 4: (Optional) Use $\Delta\mathbf{R}(\vec{\alpha})$ and $\vec{\varepsilon}$ to update an estimate for $\mathbf{F}_{LR}$ and iterate the above steps to convergence. In this case, the formulas will be slightly different since you won't be able to assume $\mathbf{R}_{LR} = \mathbf{I}$.

D. (10 points) Suppose that the value of $\mathbf{F}_{LB}$ is only known approximately. I.e.,

$$\mathbf{F}_{LB}{}^{*} = \mathbf{F}_{LB} \bullet \Delta\mathbf{F}_{LB}$$
$$\Delta\mathbf{F}_{LB} = \left[\mathbf{R}(\vec{\beta}), \vec{\gamma}\right]$$

for small $\vec{\beta}$ and $\vec{\gamma}$. Further, the values of $\mathbf{F}_{LB}$ and $\vec{\mathbf{a}}_i$ are such that all of the calibration markers are within a sphere of radius approximately $X/4$ and centered at a point approximately distance X from each of the detectors.[1] Give formulas approximating $\vec{\mathbf{b}}_i$ and $\vec{\mathbf{c}}_i$ that the elements of $\vec{\alpha}$, $\vec{\varepsilon}$, $\vec{\beta}$, and $\vec{\gamma}$ only appear linearly.

Note that now

$$\begin{aligned}
\vec{\mathbf{b}}_i &= \mathbf{F}_{LB} \cdot \Delta\mathbf{F}_{LB} \cdot \vec{\mathbf{a}}_i \\
&= \mathbf{R}_{LB} \cdot \Delta\mathbf{R}_{LB}(\vec{\beta}) \cdot \vec{\mathbf{a}}_i + \mathbf{R}_{LB} \cdot \vec{\gamma} + \vec{\mathbf{p}}_{LB} \\
&\approx \mathbf{R}_{LB} \cdot \vec{\mathbf{a}}_i + \mathbf{R}_{LB} \cdot \vec{\beta} \times \vec{\mathbf{a}} + \mathbf{R}_{LB} \cdot \vec{\gamma} + \vec{\mathbf{p}}_{LB}
\end{aligned}$$

Substituting in this expression into our equation for $\vec{\mathbf{c}}_i$ gives

$$\begin{aligned}
\vec{\mathbf{c}}_i &= \Delta\mathbf{R}(\vec{\alpha})^{-1} \cdot \vec{\mathbf{b}}_i - \Delta\mathbf{R}(\vec{\alpha})^{-1} \cdot \vec{\mathbf{p}}_{\mathbf{LR}} - \Delta\mathbf{R}(\vec{\alpha})^{-1} \cdot \vec{\varepsilon} \\
&\approx \vec{\mathbf{b}}_i - \vec{\mathbf{p}}_{\mathbf{LR}} + \vec{\alpha} \times \left(\vec{\mathbf{p}}_{\mathbf{LR}} - \vec{\mathbf{b}}_i\right) - \vec{\varepsilon} \\
&\approx \mathbf{R}_{LB} \cdot \vec{\mathbf{a}}_i + \vec{\mathbf{p}}_{LB} - \vec{\mathbf{p}}_{\mathbf{LR}} + \mathbf{R}_{LB} \cdot \vec{\beta} \times \vec{\mathbf{a}} + \mathbf{R}_{LB} \cdot \vec{\gamma} + \vec{\alpha} \times \left(\vec{\mathbf{p}}_{\mathbf{LR}} - \vec{\mathbf{b}}_i\right) - \vec{\varepsilon} \\
&\approx \mathbf{R}_{LB} \cdot \vec{\mathbf{a}}_i + \vec{\mathbf{p}}_{LB} - \vec{\mathbf{p}}_{\mathbf{LR}} + \mathbf{R}_{LB} \cdot \vec{\beta} \times \vec{\mathbf{a}} + \mathbf{R}_{LB} \cdot \vec{\gamma} + \vec{\alpha} \times \left(\vec{\mathbf{p}}_{\mathbf{LR}} - \mathbf{R}_{LB} \cdot \vec{\mathbf{a}}_i + \vec{\mathbf{p}}_{LB}\right) - \vec{\varepsilon} \\
&= \vec{\mathbf{d}}_i + \mathbf{R}_{LB} \cdot \vec{\beta} \times \vec{\mathbf{a}}_i + \mathbf{R}_{LB} \cdot \vec{\gamma} + \vec{\mathbf{d}}_i \times \vec{\alpha} - \vec{\varepsilon} \text{ where } \vec{\mathbf{d}}_i = \mathbf{R}_{LB} \cdot \vec{\mathbf{a}}_i + \vec{\mathbf{p}}_{LB} \\
&= \vec{\mathbf{d}}_i + \left(\mathbf{R}_{LB} \cdot skew(-\vec{\mathbf{a}}_i)\right) \cdot \vec{\beta} + \mathbf{R}_{LB} \cdot \vec{\gamma} + \vec{\mathbf{d}}_i \times \vec{\alpha} - \vec{\varepsilon}
\end{aligned}$$

E. (10 points) Explain how you would modify the answer to Quest ion 1.C under the assumptions in Question 1.D. Again, give all pertinent formulas.

Note that you now have a system of the form

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[1] **Note:** There is nothing particularly magical about these distances. I am just specifying a position of the calibration marker that is not likely to get you into any weirdly degenerate configurations.

$$\vec{\mathbf{b}}_i \approx \vec{\mathbf{d}}_i - \mathbf{R}_{LB} \cdot skew(\vec{\mathbf{a}}) \cdot \vec{b} + \mathbf{R}_{LB} \cdot \vec{\gamma}$$
$$\vec{\mathbf{c}}_i \approx \vec{\mathbf{d}}_i + \left(\mathbf{R}_{LB} \cdot skew(-\vec{\mathbf{a}}_i)\right) \cdot \vec{\beta} + \mathbf{R}_{LB} \cdot \vec{\gamma} + \vec{\mathbf{d}}_i \times \vec{\alpha} - \vec{\varepsilon}$$
$$u_{x,i} b_{z,i} = f b_{x,i}$$
$$u_{y,i} b_{z,i} = f b_{y,i}$$
$$v_{x,i} c_{z,i} = f c_{x,i}$$
$$v_{y,i} c_{z,i} = f c_{y,i}$$

Solve this system with linear least squares for $\vec{\alpha}, \vec{\varepsilon}, \vec{\beta},$ and $\vec{\gamma}$. Then update your estimates of $\mathbf{F}_{LB}$ and $\mathbf{F}_{LR}$ and iterate the solution.

F. (10 points) Suppose that you do not know an initial estimate for $\mathbf{F}_{LB}$, although you still know that the calibration body is located so that all of the calibration markers are within a sphere of radius approximately $X/4$ and centered at a point approximately distance X from each of the detectors. If you are given values of $\vec{\mathbf{a}}_i$, $\vec{\mathbf{u}}_i$, $\vec{\mathbf{v}}_i$, and an approximate value of X, is this sufficient information to estimate the value of $\mathbf{F}_{LB}$? If so, outline a procedure for doing so. If not, explain what additional information is needed?

The basic approach is to assume initially that $\vec{\alpha}$ and $\vec{\varepsilon}$ are $\vec{0}$. Solve

$$u_{x,i} b_{z,i} = f b_{x,i}$$
$$u_{y,i} b_{z,i} = f b_{y,i}$$
$$v_{x,i} c_{z,i} = f c_{x,i}$$
$$v_{y,i} c_{z,i} = f c_{y,i}$$
$$\vec{\mathbf{c}}_i = \mathbf{F_{LR}}^{-1} \cdot \vec{\mathbf{b}}_i$$

for $\vec{\mathbf{b}}_i$ and $\vec{\mathbf{c}}_i$. Then, use the known values of $\vec{\mathbf{a}}_i$ and your estimated values of $\vec{\mathbf{b}}_i$ to estimate for $\mathbf{F}_{LR}$. Then you can use the method in 1.E. The main thing that this requires is that the tracker be not too badly warped and that the various optimization problems not be too badly conditioned numerically.

Question 2

Consider a fluoroscopic dewarping calibration fixture similar to that discussed in class. The fixture consists of a rectangular grid of small radio-opaque spheres spaced at a distance d millimeters apart from each other. The origin $(0,0)$ is located at the center of the grid, and the individual spheres are located at points $\vec{\mathbf{f}}_{i,j} = [i \times d, j \times d, 0]$ for integers i and j. Suppose that the transformation from the detector's coordinate system to the calibration plate is given by $\mathbf{F}_{DC} = [\mathbf{R}_{DC}, \vec{\mathbf{p}}_{PC}]$.

The detector diameter is D millimeters, the detector pixels are square, and the detector pixel size is $\rho = D/1024$. In the absence of distortion, a reported pixel position of $\vec{\mathbf{u}} = [u_x, u_y]$ would correspond to a physical position $\vec{\mathbf{h}} = \rho \vec{\mathbf{u}}$ on the detector. However the fluoroscope has an unknown distortion

$\vec{\mathbf{g}}(\vec{\mathbf{u}}) = \left[g_x(\vec{\mathbf{u}}), g_y(\vec{\mathbf{u}}) \right]$ such that the actual position on the detector corresponding to a detected position $\vec{\mathbf{u}}$ would be $\vec{\mathbf{h}}^* = \rho\left(\vec{\mathbf{u}} + \vec{\mathbf{g}}(\vec{\mathbf{u}})\right)$. Although $\vec{\mathbf{g}}(\vec{\mathbf{u}})$ is unknown, we know that it does not change quickly.

$$\left|\frac{\delta g_x}{\delta u_x}\right| \le \gamma_{x,x} \quad \left|\frac{\delta g_x}{\delta u_y}\right| \le \gamma_{x,y}$$
$$\left|\frac{\delta g_y}{\delta u_x}\right| \le \gamma_{y,x} \quad \left|\frac{\delta g_y}{\delta u_y}\right| \le \gamma_{x,y}$$

Suppose that the dewarp calibration fixture is placed directly on top of the detector (i.e., suppose that the transformation from the fixture plate to detector is the identity, $\mathbf{F}_{DC} = \mathbf{I}$) and that the computer is capable of determining the coordinates $\vec{\mathbf{u}}_{i,j}$ corresponding to the apparent image of each fiducial point $\vec{\mathbf{f}}_{i,j}$.

A. 15 points) Outline a simple computational procedure for estimating the distortion $\vec{\mathbf{g}}^{(est)}(\vec{\mathbf{u}}) \approx \vec{\mathbf{g}}(\vec{\mathbf{u}})$ using bilinear interpolation and show how this can be applied to find an estimate $\vec{\mathbf{h}}^{(est)}(\vec{\mathbf{u}}) \approx \vec{\mathbf{h}}(\vec{\mathbf{u}})$. Include all relevant formulas.

Let $\vec{\mathbf{u}}_{i,j}$ be the position in detector coordinates reported for the $(i,j)^{\text{th}}$ fiducial. We then know the following relationship

$$\vec{\mathbf{f}}_{i,j} = \rho\vec{\mathbf{u}}_{i,j} + \rho\vec{\mathbf{g}}(\vec{\mathbf{u}}_{i,j})$$

Step 0: Define

$$\vec{\mathbf{g}}_{i,j} = \frac{\vec{\mathbf{f}}_{i,j}}{\rho} - \vec{\mathbf{u}}_{i,j}$$

Step 1: Given an arbitrary value $\vec{\mathbf{u}}$ find an (i,j) and (λ,μ) such that

$$\begin{aligned}
&\vec{\mathbf{u}} = (1-\mu)\left((1-\lambda)\vec{\mathbf{u}}_{i,j} + \lambda\vec{\mathbf{u}}_{i+1,j}\right) + \mu\left((1-\lambda)\vec{\mathbf{u}}_{i,j+1} + \lambda\vec{\mathbf{u}}_{i+1,j+1}\right)\\
&0 \le \lambda \le 1\\
&0 \le \mu \le 1
\end{aligned}$$

Step 2: Compute

$$\begin{aligned}
&\vec{\mathbf{g}}^{(est)}(\vec{\mathbf{u}}) \approx (1-\mu)\left((1-\lambda)\vec{\mathbf{g}}_{i,j} + \lambda\vec{\mathbf{g}}_{i+1,j}\right) + \mu\left((1-\lambda)\vec{\mathbf{g}}_{i,j+1} + \lambda\vec{\mathbf{g}}_{i+1,j+1}\right)\\
&\vec{\mathbf{h}}^{(est)}(\vec{\mathbf{u}}) \approx \rho\left(\vec{\mathbf{u}} + \vec{\mathbf{g}}^{(est)}(\vec{\mathbf{u}})\right)
\end{aligned}$$

B. (20 points) For the procedure given in A, estimate the maximum error. I.e., if

$$\vec{\mathbf{h}}^{(err)}(\vec{\mathbf{u}}) = \vec{\mathbf{h}}^*(\vec{\mathbf{u}}) - \vec{\mathbf{h}}^{(est)}(\vec{\mathbf{u}})$$

then produce formulas giving bounds on $h_x^{(err)}(\vec{\mathbf{u}})$ and $h_y^{(err)}(\vec{\mathbf{u}})$.

For simplicity, let's define

$$\begin{aligned}\vec{\mathbf{a}} &= \vec{\mathbf{u}}_{i+1,j} - \vec{\mathbf{u}}_{i,j}\\ \vec{\mathbf{b}} &= \vec{\mathbf{u}}_{i,j+1} - \vec{\mathbf{u}}_{i,j}\\ \vec{\mathbf{c}} &= \left(\vec{\mathbf{u}}_{i+1,j+1} + \vec{\mathbf{u}}_{i,j}\right) - \left(\vec{\mathbf{u}}_{i+1,j} + \vec{\mathbf{u}}_{i,j+1}\right)\end{aligned}$$

Then,

$$\vec{\mathbf{u}}(\lambda,\mu) = \vec{\mathbf{u}}_{i,j} + \lambda\vec{\mathbf{a}} + \mu\vec{\mathbf{b}} + \lambda\mu\vec{\mathbf{c}}$$

Similarly, we can define

$$\begin{aligned}\vec{A} &= \vec{\mathbf{g}}_{i+1,j} - \vec{\mathbf{g}}_{i,j}\\ \vec{B} &= \vec{\mathbf{g}}_{i,j+1} - \vec{\mathbf{g}}_{i,j}\\ \vec{C} &= \left(\vec{\mathbf{g}}_{i+1,j+1} + \vec{\mathbf{g}}_{i,j}\right) - \left(\vec{\mathbf{g}}_{i+1,j} + \vec{\mathbf{g}}_{i,j+1}\right)\end{aligned}$$

and define

$$\begin{aligned}\vec{G}(\lambda,\mu) &= \vec{\mathbf{g}}(\vec{\mathbf{u}}_{i,j} + \lambda\vec{\mathbf{a}} + \mu\vec{\mathbf{b}} + \lambda\mu\vec{\mathbf{c}})\\ \vec{G}^{est}(\lambda,\mu) &= \vec{\mathbf{g}}_{i,j} + \lambda\vec{A} + \mu\vec{B} + \lambda\mu\vec{C}\end{aligned}$$

Further,

$$\begin{aligned}\frac{\partial\vec{G}(\lambda,\mu)}{\partial\lambda} &= \frac{\partial\vec{\mathbf{g}}}{\partial u_x}\frac{\partial u_x}{\partial\lambda} + \frac{\partial\vec{\mathbf{g}}}{\partial u_y}\frac{\partial u_y}{\partial\lambda}\\ &= \frac{\partial\vec{\mathbf{g}}}{\partial u_x}\left(a_x + \mu c_x\right) + \frac{\partial\vec{\mathbf{g}}}{\partial u_y}\left(a_y + \mu c_y\right)\end{aligned}$$

$$\left|\frac{\partial\vec{G}(\lambda,\mu)}{\partial\lambda}\right| \le \begin{bmatrix}\gamma_{x,x}\left(|a_x| + \mu|c_x|\right) + \gamma_{x,y}\left(|a_y| + \mu|c_y|\right)\\ \gamma_{y,x}\left(|a_x| + \mu|c_x|\right) + \gamma_{y,y}\left(|a_y| + \mu|c_y|\right)\end{bmatrix}$$

$$\left|\frac{\partial\vec{G}(\lambda,\mu)}{\partial\lambda}\right| \le \begin{bmatrix}\gamma_{x,x}|a_x| + \gamma_{x,y}|a_y|\\ \gamma_{y,x}|a_x| + \gamma_{y,y}|a_y|\end{bmatrix} + \mu\begin{bmatrix}\gamma_{x,x}|c_x| + \gamma_{x,y}|c_y|\\ \gamma_{y,x}|c_x| + \gamma_{y,y}|c_y|\end{bmatrix}$$

and

$$\begin{aligned}\frac{\partial\vec{G}(\lambda,\mu)}{\partial\mu} &= \frac{\partial\vec{\mathbf{g}}}{\partial u_x}\frac{\partial u_x}{\partial\mu} + \frac{\partial\vec{\mathbf{g}}}{\partial u_y}\frac{\partial u_y}{\partial\mu}\\ &= \frac{\partial\vec{\mathbf{g}}}{\partial u_x}\left(b_x + \lambda c_x\right) + \frac{\partial\vec{\mathbf{g}}}{\partial u_y}\left(b_y + \lambda c_y\right)\end{aligned}$$

$$\left|\frac{\partial\vec{G}(\lambda,\mu)}{\partial\mu}\right| \le \begin{bmatrix}\gamma_{x,x}|b_x| + \gamma_{x,y}|b_y|\\ \gamma_{y,x}|b_x| + \gamma_{y,y}|b_y|\end{bmatrix} + \lambda\begin{bmatrix}\gamma_{x,x}|c_x| + \gamma_{x,y}|c_y|\\ \gamma_{y,x}|c_x| + \gamma_{y,y}|c_y|\end{bmatrix}$$

We can express the error in our distortion approximation as a function of (λ,μ) as well:

$$
\begin{aligned}
\vec{H}^{err}(\lambda,\mu) &= \vec{\mathbf{h}}^{err}(\vec{\mathbf{u}}_{i,j} + \lambda\vec{\mathbf{a}} + \mu\vec{\mathbf{b}} + \lambda\mu\vec{\mathbf{c}}) \\
&= \rho\vec{G}(\lambda,\mu) - \rho\vec{G}^{est}(\lambda,\mu) \\
&= \rho\vec{G}(\lambda,\mu) - \rho\vec{G}(0,0) - \rho\lambda\vec{A} - \rho\mu\vec{B} - \rho\lambda\mu\vec{C}
\end{aligned}
$$

Note that

$$\vec{H}^{err}(0,0) = \vec{H}^{err}(1,0) = \vec{H}^{err}(1,1) = \vec{H}^{err}(0,1) = 0$$

Now note that

$$
\begin{aligned}
\nabla\vec{H}^{err}(\lambda,\mu) &= \rho\nabla\vec{G}(\lambda,\mu) - \rho\nabla\vec{G}^{est}(\lambda,\mu) \\
\partial\vec{H}^{err}/\partial\lambda &= \rho\partial\vec{G}(\lambda,\mu)/\partial\lambda - \rho\vec{A} - \rho\mu\vec{C} \\
\partial\vec{H}^{err}/\partial\mu &= \rho\partial\vec{G}(\lambda,\mu)/\partial\mu - \rho\vec{B} - \rho\lambda\vec{C}
\end{aligned}
$$

Now, the error at any point

$$
\left|\vec{H}^{err}(\lambda,\mu)\right| \le \begin{cases}
\lambda\max\left|\partial\vec{H}^{err}/\partial\lambda\right| + \mu\max\left|\partial\vec{H}^{err}/\partial\mu\right| \\
(1-\lambda)\max\left|\partial\vec{H}^{err}/\partial\lambda\right| + \mu\max\left|\partial\vec{H}^{err}/\partial\mu\right| \\
\lambda\max\left|\partial\vec{H}^{err}/\partial\lambda\right| + (1-\mu)\max\left|\partial\vec{H}^{err}/\partial\mu\right| \\
(1-\lambda)\max\left|\partial\vec{H}^{err}/\partial\lambda\right| + (1-\mu)\max\left|\partial\vec{H}^{err}/\partial\mu\right|
\end{cases}
$$

Now

$$
\begin{aligned}
\left|\partial\vec{H}^{err}/\partial\lambda\right| &= \rho\left|\partial\vec{G}(\lambda,\mu)/\partial\lambda - \vec{A} - \mu\vec{C}\right| \\
&\le \rho\begin{bmatrix} \gamma_{x,x}|a_x| + \gamma_{x,y}|a_y| + |A_x| \\ \gamma_{y,x}|a_x| + \gamma_{y,y}|a_y| + |A_y| \end{bmatrix} + \mu\rho\begin{bmatrix} \gamma_{x,x}|c_x| + \gamma_{x,y}|c_y| + |C_x| \\ \gamma_{y,x}|c_x| + \gamma_{y,y}|c_y| + |C_y| \end{bmatrix} \\
\left|\partial\vec{H}^{err}/\partial\mu\right| &= \left|\rho\partial\vec{G}(\lambda,\mu)/\partial\mu - \rho\vec{B} - \rho\lambda\vec{C}\right| \\
&\le \rho\begin{bmatrix} \gamma_{x,x}|b_x| + \gamma_{x,y}|b_y| + |B_x| \\ \gamma_{y,x}|b_x| + \gamma_{y,y}|b_y| + |B_y| \end{bmatrix} + \lambda\rho\begin{bmatrix} \gamma_{x,x}|c_x| + \gamma_{x,y}|c_y| + |C_x| \\ \gamma_{y,x}|c_x| + \gamma_{y,y}|c_y| + |C_y| \end{bmatrix}
\end{aligned}
$$

We can rewrite this as

$$
\begin{aligned}
\left|\partial\vec{H}^{err}/\partial\lambda\right| &\le \rho\begin{bmatrix} P_x \\ P_y \end{bmatrix} + \mu\rho\begin{bmatrix} R_x \\ R_y \end{bmatrix} \\
\left|\partial\vec{H}^{err}/\partial\mu\right| &\le \rho\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} + \lambda\rho\begin{bmatrix} R_x \\ R_y \end{bmatrix}
\end{aligned}
$$

So

$$\left|\vec{H}^{err}(\lambda,\mu)\right| \le \begin{cases} \lambda\left(\rho\begin{bmatrix}P_x\\P_y\end{bmatrix}+\mu\rho\begin{bmatrix}R_x\\R_y\end{bmatrix}\right)+\mu\left(\rho\begin{bmatrix}Q_x\\Q_y\end{bmatrix}+\lambda\rho\begin{bmatrix}R_x\\R_y\end{bmatrix}\right) \\ (1-\lambda)\left(\rho\begin{bmatrix}P_x\\P_y\end{bmatrix}+\mu\rho\begin{bmatrix}R_x\\R_y\end{bmatrix}\right)+\mu\left(\rho\begin{bmatrix}Q_x\\Q_y\end{bmatrix}+\lambda\rho\begin{bmatrix}R_x\\R_y\end{bmatrix}\right) \\ \lambda\left(\rho\begin{bmatrix}P_x\\P_y\end{bmatrix}+\mu\rho\begin{bmatrix}R_x\\R_y\end{bmatrix}\right)+(1-\mu)\left(\rho\begin{bmatrix}Q_x\\Q_y\end{bmatrix}+\lambda\rho\begin{bmatrix}R_x\\R_y\end{bmatrix}\right) \\ (1-\lambda)\left(\rho\begin{bmatrix}P_x\\P_y\end{bmatrix}+\mu\rho\begin{bmatrix}R_x\\R_y\end{bmatrix}\right)+\left(\rho\begin{bmatrix}Q_x\\Q_y\end{bmatrix}+\lambda\rho\begin{bmatrix}R_x\\R_y\end{bmatrix}\right) \end{cases}$$

For $0 \le \lambda,\mu \le 1$. This can clean up a bit to

$$\left|\vec{H}^{err}(\lambda,\mu)\right| \le \begin{cases} \rho\left(\lambda\vec{P}+\mu\vec{Q}+2\lambda\mu\vec{R}\right) \\ \rho\left((1-\lambda)\vec{P}+\mu\left(\vec{Q}+\vec{R}\right)\right) \\ \rho\left(\lambda\left(\vec{P}+\vec{R}\right)+(1-\mu)\vec{Q}\right) \\ \rho\left(\left(\vec{P}+\vec{Q}\right)+\lambda\left(\vec{R}-\vec{P}\right)+\mu\left(\vec{R}-\vec{P}\right)-2\lambda\mu\vec{R}\right) \end{cases}$$

Now, one could in principle solve these of these inequalities over (λ,μ) to find a tight bound.

$\left|\vec{H}^{err}(\lambda,\mu)\right| \le \eta^*$ where

$\eta^* = \arg\min_{(\lambda,\mu)} \eta$ subject to

$$\eta \le \begin{cases} \rho\left(\lambda\vec{P}+\mu\vec{Q}+2\lambda\mu\vec{R}\right) \\ \rho\left((1-\lambda)\vec{P}+\mu\left(\vec{Q}+\vec{R}\right)\right) \\ \rho\left(\lambda\left(\vec{P}+\vec{R}\right)+(1-\mu)\vec{Q}\right) \\ \rho\left(\left(\vec{P}+\vec{Q}\right)+\lambda\left(\vec{R}-\vec{P}\right)+\mu\left(\vec{R}-\vec{P}\right)-2\lambda\mu\vec{R}\right) \end{cases}$$

But a reasonable, perhaps slightly looser, bound can be found by evaluating this at $\lambda=\mu=\frac{1}{2}$ and taking $\eta^*=\rho\left(\vec{P}+\vec{Q}+\vec{R}\right)/2$

C. 10 points) Suppose now that the $\vec{\mathbf{u}}_{i,j}$ are subject to some measurement errors, so that

$$\vec{\mathbf{u}}_{i,j}{}^{*} = \vec{\mathbf{u}}_{i,j} + \Delta\vec{\mathbf{u}}_{i,j}$$

$$\Delta\vec{\mathbf{u}}_{i,j} \approx \left[\mu_{i,j}, \nu_{i,j}\right] \text{ where } \left|\mu_{i,j}\right| < \mu^{\max}, \ \left|\nu_{i,j}\right| < \nu^{\max} \quad \text{(small numbers)}$$

Produce a revised formula estimating $\vec{\mathbf{h}}^{(err)}(\vec{\mathbf{u}}) = \vec{\mathbf{h}}^{*}(\vec{\mathbf{u}}) - \vec{\mathbf{h}}^{(est)}(\vec{\mathbf{u}})$.

One can in principle grind through formulas very similar to those in the previous part, adding in $\mu^{\max}$ and $\nu^{\max}$ at appropriate places. However, if we assume that the distortion doesn't change fast and that the measurement errors are small, then the following estimate of the distortion is close enough

$$\begin{aligned}
\left|\vec{H}^{err}(\lambda,\mu)\right| &\le \rho\left(\vec{N} + \lambda\vec{P} + \mu\vec{Q} + 2\lambda\mu\vec{R}\right) \\
&\le \rho\left(\vec{N} + (1-\lambda)\vec{P} + \mu\left(\vec{Q} + \vec{R}\right)\right) \\
&\le \rho\left(\vec{N} + \lambda\left(\vec{P} + \vec{R}\right) + (1-\mu)\vec{Q}\right) \\
&\le \rho\left(\vec{N} + \left(\vec{P} + \vec{Q}\right) + \lambda\left(\vec{R} - \vec{P}\right) + \mu\left(\vec{R} - \vec{P}\right) - 2\lambda\mu\vec{R}\right)
\end{aligned}$$

where

$$\vec{N} = \begin{bmatrix} \mu^{\max} \\ \nu^{\max} \end{bmatrix}$$