Homework Assignment 3 – 600.445 Fall 2004

Instructions and Score Sheet (hand in with answers)

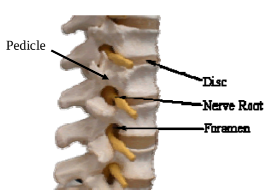
Name	
Email	
Other contact information (optional)	
Signature (required)	I have followed the rules in completing this assignment

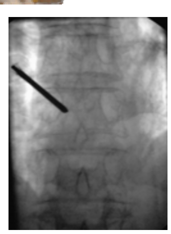
Question	Points	Points
1A	3	
1B	3	
1C	3	
1D	3	
1E	8	
2	20	
3A	5	
3B	5	
3C	5	
3D	5	
4A	15	
4B	5	
5A	10	
5B	10	
6	15	
Total	115	

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
6. Sign and hand in the score sheet as the first sheet of your assignment.
7. Remember to include a sealable 8 ½ by 11 inch self-addressed envelope if you want your assignment

Computer-assisted vertebroplasty

- Injection of cement into vertebral bodies using large needle inserted through pedicle
- Indications include fractures, osteoporosis, tumors
- Often relieves pain
- Typically performed freehand under x-ray guidance
- For current problem, we will assume a navigation assisted method
- Note: could equally have assumed pedicle screw placement for this problem





Problem Scenario: computer assisted trans-pedicle insertion

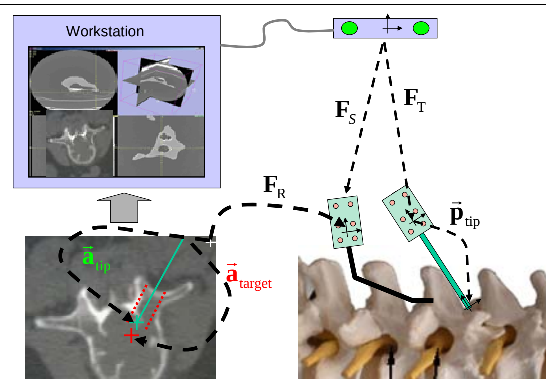


Figure 1: Computer-assisted trans-pedicle shaft insertion

Consider the computer-assisted vertebroplasty or pedicle screw insertion system illustrated in Figure 1. The goal is to insert a navigated shaft down the patient's pedicle into the vertebral body without going outside the cortical wall of the pedicle (and so hitting something like the spinal cord, a nerve root, etc.).

A reference body S is attached to the spine. A second reference body T is attached to a pointer, whose tip is nominally located at a nominal offset $\vec{\mathbf{p}}_{\text{tip}}$ relative to the coordinate system of reference body T. Assume that we have the following other relationships:

$\mathbf{F}_{\text{T}} = [\mathbf{R}_{\text{T}}, \vec{\mathbf{p}}_{\text{T}}]$ is the reported pose of body T relative to the tracker

$\mathbf{F}_{S} = [\mathbf{R}_{\text{S}}, \vec{\mathbf{p}}_{\text{S}}]$ is the reported pose of body S relative to the tracker

$\mathbf{F}_{R} = [\mathbf{R}_{\text{R}}, \vec{\mathbf{p}}_{\text{R}}]$ is the registration transformation between S and CT

$\vec{\mathbf{a}}_{\text{target}}$ = the position in CT coordinates of a target point

$\vec{\mathbf{a}}_{\text{tip}}$ = the position in CT coordinates of the pointer tip

Thus, if $\vec{\mathbf{v}}$ is the position of a point relative to tracker body S, the corresponding point will have CT coordinates $\mathbf{F}_{R} \bullet \vec{\mathbf{v}}$.

We will assume that the tool shaft axis passes through the origin of the coordinate system of body T.

You may also assume that the nominal value of $\vec{\mathbf{p}}_{\text{tip}}$ is $\lambda\vec{\mathbf{z}} = [0,0,\lambda]^{T}$.

Suppose, further that surgical planning software has identified a "safe path" through the pedicle as a cylindrical tube of radius r_{safe} centered on a line segment between points whose CT coordinates are $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$. The path is asserted to be safe so long as the centerline of the tool enters and exits the tube through its ends and not through its sides.

Question 1: Basic Geometric Relationships

A. Give an expression for computing the position $\vec{\mathbf{p}}_{St}$ of the pointer relative to reference body S in terms of $\vec{\mathbf{p}}_{\text{tip}}$ and the components of $\mathbf{F}_{\text{S}}$ and $\mathbf{F}_{T}$ (i.e., $\mathbf{R}_{S}, \vec{\mathbf{p}}_{S}$, etc.)

$$\begin{aligned}\vec{\mathbf{p}}_{St} &= \mathbf{F}_{S}^{-1}\mathbf{F}_{T}\vec{\mathbf{p}}_{tip} \\ &= \left[\mathbf{R}_{S}^{-1}, -\vec{\mathbf{p}}_{S}\right] \bullet \left(\mathbf{R}_{T}\vec{\mathbf{p}}_{tip} + \vec{\mathbf{p}}_{T}\right) \\ &= \mathbf{R}_{S}^{-1}\mathbf{R}_{T}\vec{\mathbf{p}}_{tip} + \mathbf{R}_{S}^{-1}\vec{\mathbf{p}}_{T} - \mathbf{R}_{S}^{-1}\vec{\mathbf{p}}_{S}\end{aligned}$$

B. Give an expression the workstation can use for computing the value $\vec{\mathbf{e}}_{\text{tip}} = \vec{\mathbf{a}}_{\text{tip}} - \vec{\mathbf{a}}_{\text{target}}$.

$$\begin{aligned}\vec{\mathbf{a}}_{tip} &= \mathbf{F}_{R}\mathbf{F}_{S}^{-1}\mathbf{F}_{T}\vec{\mathbf{p}}_{tip} \\ &= \mathbf{R}_{R}\mathbf{R}_{S}^{-1}\mathbf{R}_{T}\vec{\mathbf{p}}_{tip} + \mathbf{R}_{R}\mathbf{R}_{S}^{-1}\vec{\mathbf{p}}_{T} - \mathbf{R}_{R}\mathbf{R}_{S}^{-1}\vec{\mathbf{p}}_{S} + \vec{\mathbf{p}}_{R} \\ \vec{\mathbf{e}}_{tip} &= \vec{\mathbf{a}}_{tip} - \vec{\mathbf{a}}_{\text{target}} \\ &= \mathbf{R}_{R}\mathbf{R}_{S}^{-1}\mathbf{R}_{T}\vec{\mathbf{p}}_{tip} + \mathbf{R}_{R}\mathbf{R}_{S}^{-1}\vec{\mathbf{p}}_{T} - \mathbf{R}_{R}\mathbf{R}_{S}^{-1}\vec{\mathbf{p}}_{S} + \vec{\mathbf{p}}_{R} - \vec{\mathbf{a}}_{\text{target}}\end{aligned}$$

C. Suppose that we want to assure that the tip of the tool is inside safe path tube. Give a set of geometric constraints that $\vec{\mathbf{a}}_{\text{tip}}$ must obey if this is true.

Define

$$\vec{\mathbf{n}}_{12} = \frac{\vec{\mathbf{a}}_2 - \vec{\mathbf{a}}_1}{d_{12}} \qquad \text{where } d_{12} = \left\|\vec{\mathbf{a}}_2 - \vec{\mathbf{a}}_1\right\|$$

$$\delta = \left(\vec{\mathbf{a}}_{tip} - \vec{\mathbf{a}}_1\right) \bullet \vec{\mathbf{n}}_{12}$$

$$\mathbf{b} = \vec{\mathbf{a}}_{tip} - \vec{\mathbf{a}}_1 - \delta\vec{\mathbf{n}}_{12}$$

The constraints will be

$$0 \le \delta \le d_{21}$$

$$\left\|\vec{\mathbf{b}}\right\| \le r_{\text{safe}}$$

Note: we will give almost full credit if you leave off the λ constraint. If linearized constraints are desired, one can compute a simple version just by requiring that the components of $\mathbf{R}_{nz} \bullet \left(\vec{\mathbf{a}}_{tip} - \vec{\mathbf{a}}_1\right)$ be less than $r_{safe} / \sqrt{2}$, where $\mathbf{R}_{nz}$ is some rotation that rotates $\vec{\mathbf{n}}_{\mathbf{12}}$ to the $\vec{\mathbf{z}}$ axis. An alternative set of non-linearized constraints would be

$$\vec{\mathbf{d}} = \mathbf{R}_{nz} \bullet \left(\vec{\mathbf{a}}_{tip} - \vec{\mathbf{a}}_1\right) = \left[d_x, d_y, d_z\right]$$
$$0 \le d_z \le d_{12}$$
$$d_x^{\ 2} + d_y^{\ 2} \le r_{\text{safe}}^{\ 2} \quad \text{or the linearized equivalent}$$

D. Express these constraints in terms of $\vec{\mathbf{p}}_{\text{tip}}$ and the components of $\mathbf{F}_S$, $\mathbf{F}_R$ and $\mathbf{F}_T$.

$$\vec{\mathbf{d}} = \mathbf{R}_{nz} \bullet \left(\vec{\mathbf{a}}_{tip} - \vec{\mathbf{a}}_1\right) = \left[d_x, d_y, d_z\right]$$
$$\vec{\mathbf{d}} = \mathbf{R}_{nz} \bullet \left(\mathbf{R}_R \mathbf{R}_S^{-1} \mathbf{R}_T \vec{\mathbf{p}}_{tip} + \mathbf{R}_R \mathbf{R}_S^{-1} \vec{\mathbf{p}}_T - \mathbf{R}_R \mathbf{R}_S^{-1} \vec{\mathbf{p}}_S + \vec{\mathbf{p}}_R - \vec{\mathbf{a}}_1\right)$$
$$0 \le d_z \le d_{12}$$
$$d_x^{\ 2} + d_y^{\ 2} \le r_{\text{safe}}^{\ 2} \quad \text{or the linearized equivalent}$$

E. Give constraint expressions in terms of $\vec{\mathbf{p}}_{\text{tip}}$ and the components of $\mathbf{F}_S$, $\mathbf{F}_R$ and $\mathbf{F}_T$ that must be obeyed if the tool shaft is to remain inside the safety tube while the tip is inserted through the pedicle.

The thing to realize is that you already have most of the answer from the previous question. Without loss of generality, assume that $\vec{\mathbf{a}}_1$ is the entrance to the safe tube (from outside the vertebra) and $\vec{\mathbf{a}}_2$ is the exit into the vertebral body. If $\vec{\mathbf{a}}_{tip}$ is inside the tube, then we want to compute the intersection of the pointer shaft with the plane perpendicular to $\vec{\mathbf{n}}_{12}$ that passes through $\vec{\mathbf{a}}_1$. Any point on the pointer shaft (in CT coordinates) may be represented by

$$\vec{\mathbf{a}}_{shaft} = \vec{\mathbf{a}}_{tip} - \mu \vec{\mathbf{n}}_{shaft}$$
$$\mu \ge 0$$
$$\vec{\mathbf{n}}_{shaft} = \mathbf{R}_R \mathbf{R}_S^{-1} \mathbf{R}_T \vec{\mathbf{z}}$$

Any point $\vec{\mathbf{a}}$ on the plane in question must obey

$$\left(\vec{\mathbf{a}} - \vec{\mathbf{a}}_1\right) \bullet \vec{\mathbf{n}}_{12} = 0$$

Substituting the definition of $\vec{\mathbf{a}}_{shaft}$ will enable us to solve for μ

$$0 = \left(\vec{\mathbf{a}}_{tip} - \vec{\mathbf{a}}_1 - \mu \vec{\mathbf{n}}_{shaft}\right) \bullet \vec{\mathbf{n}}_{12}$$
$$\mu = \frac{\vec{\mathbf{n}}_{12} \bullet \left(\vec{\mathbf{a}}_{tip} - \vec{\mathbf{a}}_1\right)}{\vec{\mathbf{n}}_{12} \bullet \vec{\mathbf{n}}_{shaft}}$$
$$\mu = \frac{\vec{\mathbf{n}}_{12} \bullet \left(\mathbf{R}_R \mathbf{R}_S^{-1} \mathbf{R}_T \vec{\mathbf{p}}_{tip} + \mathbf{R}_R \mathbf{R}_S^{-1} \vec{\mathbf{p}}_{tip} - \mathbf{R}_R \vec{\mathbf{p}}_S + \vec{\mathbf{p}}_R - \vec{\mathbf{a}}_1\right)}{\vec{\mathbf{n}}_{12} \bullet \vec{\mathbf{n}}_{shaft}}$$

Now, we can substitute this value of μ to find the point $\vec{\mathbf{a}}^1_{shaft}$ where the shaft crosses the plane.

$$\vec{\mathbf{a}}^1_{shaft} = \vec{\mathbf{a}}_{tip} - \mu \vec{\mathbf{n}}_{shaft}$$
$$= \vec{\mathbf{a}}_{tip} - \mu \mathbf{R}_R \mathbf{R}_S^{-1} \mathbf{R}_T \vec{\mathbf{z}}$$

Our second constraint is now

$$\left\| \vec{\mathbf{a}}^1_{shaft} - \vec{\mathbf{a}}_1 \right\| \le r_{safe}$$

I.e.,

$$\left\| \mathbf{R}_R \mathbf{R}_S^{-1} \mathbf{R}_T \vec{\mathbf{p}}_{tip} + \mathbf{R}_R \mathbf{R}_S^{-1} \vec{\mathbf{p}}_{tip} - \mathbf{R}_R \vec{\mathbf{p}}_S + \vec{\mathbf{p}}_R - \mu \mathbf{R}_R \mathbf{R}_S^{-1} \mathbf{R}_T \vec{\mathbf{z}} - \vec{\mathbf{a}}_1 \right\| \le r_{safe}$$

Question 2: Simple Registration

Suppose that three vertebral landmarks have been identified on the CT images. Let the positions of these landmarks in CT coordinates be given by $\vec{\mathbf{b}}_1$, $\vec{\mathbf{b}}_2$, and $\vec{\mathbf{b}}_3$. Outline a process for computing the registration transformation $\mathbf{F}_R$. Give formulas.

Here, I won't go into great detail. The essential steps are:

1. Touch the three landmarks & compute the corresponding positions $\vec{\mathbf{c}}_1, \vec{\mathbf{c}}_2, \vec{\mathbf{c}}_3$ in reference body coordinates
2. Use any point-cloud to point-cloud method to compute $\mathbf{F}_R$ such that $\vec{\mathbf{b}}_k = \mathbf{F}_R \bullet \vec{\mathbf{c}}_k$

Question 3: Simple Error Analysis

Suppose that the actual values of our transformations are given by

$$\mathbf{F}_S^* = \mathbf{F}_S \bullet \Delta\mathbf{F}_S$$
$$\mathbf{F}_T^* = \mathbf{F}_T \bullet \Delta\mathbf{F}_T$$
$$\mathbf{F}_R^* = \mathbf{F}_R \bullet \Delta\mathbf{F}_R$$

where $\Delta\mathbf{F}_k = [\Delta\mathbf{R}_k, \Delta\vec{\mathbf{p}}_k]$, etc. and we have the following relationships

$$\Delta\vec{\mathbf{p}}_k = \vec{\varepsilon}_k$$
$$\Delta R_k \approx \mathbf{I} + skew(\vec{\alpha}_k)$$

Likewise, suppose that the actual value of the pointer tip is

$$\vec{\mathbf{p}}_{tip}^{\;*} = \vec{\mathbf{p}}_{tip} + \Delta\vec{\mathbf{p}}_{tip} = \vec{\mathbf{p}}_{tip} + \vec{\varepsilon}_{tip}$$

A. Give an "exact expression" (in terms of the $\Delta\mathbf{R}_k$'s, etc.) for the error in the value of $\vec{\mathbf{p}}_{St}$.

The nominal expression for $\vec{\mathbf{p}}_{St}$ is

$$\begin{aligned}
\vec{\mathbf{p}}_{St} &= \mathbf{F}_S^{-1}\mathbf{F}_T\vec{\mathbf{p}}_{tip} = \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\mathbf{p}}_{tip} + \mathbf{R}_S^{-1}\vec{\mathbf{p}}_T - \mathbf{R}_S^{-1}\vec{\mathbf{p}}_S \\
&= \mathbf{R}_S^{-1}\left(\mathbf{R}_T\vec{\mathbf{p}}_{tip} + \vec{\mathbf{p}}_T - \vec{\mathbf{p}}_S\right) \\
\vec{\mathbf{p}}_{St}^{*} &= \Delta\mathbf{F}_S^{-1}\mathbf{F}_S^{-1}\mathbf{F}_T\Delta\mathbf{F}_T\vec{\mathbf{p}}_{tip} \\
&= \Delta\mathbf{F}_S^{-1}\mathbf{F}_S^{-1}\mathbf{F}_T\left(\Delta\mathbf{R}_T\vec{\mathbf{p}}_{tip} + \vec{\varepsilon}_T\right) \\
&= \Delta\mathbf{F}_S^{-1}\mathbf{F}_S^{-1}\left(\mathbf{R}_T\Delta\mathbf{R}_T\vec{\mathbf{p}}_{tip} + \mathbf{R}_T\vec{\varepsilon}_T + \vec{\mathbf{p}}_T\right) \\
&= \Delta\mathbf{F}_S^{-1}\left(\mathbf{R}_S^{-1}\mathbf{R}_T\Delta\mathbf{R}_T\vec{\mathbf{p}}_{tip} + \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\varepsilon}_T + \mathbf{R}_S^{-1}\vec{\mathbf{p}}_T - \mathbf{R}_S^{-1}\vec{\mathbf{p}}_S\right) \\
&= \begin{pmatrix} \Delta\mathbf{R}_S^{-1}\mathbf{R}_S^{-1}\mathbf{R}_T\Delta\mathbf{R}_T\vec{\mathbf{p}}_{tip} + \Delta\mathbf{R}_S^{-1}\mathbf{R}_S^{-1}\mathbf{R}_T\vec{\varepsilon}_T \\ + \Delta\mathbf{R}_S^{-1}\mathbf{R}_S^{-1}\vec{\mathbf{p}}_T - \Delta\mathbf{R}_S^{-1}\mathbf{R}_S^{-1}\vec{\mathbf{p}}_S - \Delta\mathbf{R}_S^{-1}\vec{\varepsilon}_S \end{pmatrix} \\
\vec{\varepsilon}_{St} &= \begin{pmatrix} \Delta\mathbf{R}_S^{-1}\mathbf{R}_S^{-1}\mathbf{R}_T\Delta\mathbf{R}_T\vec{\mathbf{p}}_{tip} + \Delta\mathbf{R}_S^{-1}\mathbf{R}_S^{-1}\mathbf{R}_T\vec{\varepsilon}_T \\ + \Delta\mathbf{R}_S^{-1}\mathbf{R}_S^{-1}\vec{\mathbf{p}}_T - \Delta\mathbf{R}_S^{-1}\mathbf{R}_S^{-1}\vec{\mathbf{p}}_S - \Delta\mathbf{R}_S^{-1}\vec{\varepsilon}_S \\ - \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\mathbf{p}}_{tip} - \mathbf{R}_S^{-1}\vec{\mathbf{p}}_T + \mathbf{R}_S^{-1}\vec{\mathbf{p}}_S \end{pmatrix}
\end{aligned}$$

B. Give the corresponding linearized expression in terms of the $\vec{\alpha}_k$'s.

$$\begin{aligned}
\vec{\mathbf{p}}_{St} &= \mathbf{F}_S^{-1}\mathbf{F}_T\vec{\mathbf{p}}_{tip} = \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\mathbf{p}}_{tip} + \mathbf{R}_S^{-1}\left(\vec{\mathbf{p}}_T - \vec{\mathbf{p}}_S\right) \\
\vec{\mathbf{p}}_{St}^{*} &= \Delta\mathbf{F}_S^{-1}\mathbf{F}_S^{-1}\mathbf{F}_T\Delta\mathbf{F}_T\vec{\mathbf{p}}_{tip} \\
&\approx \Delta\mathbf{F}_S^{-1}\mathbf{F}_S^{-1}\mathbf{F}_T\left(\vec{\mathbf{p}}_{tip} + \vec{\alpha}_T \times \vec{\mathbf{p}}_{tip} + \vec{\varepsilon}_T\right) \\
&= \Delta\mathbf{F}_S^{-1}\mathbf{F}_S^{-1}\left(\mathbf{R}_T\vec{\mathbf{p}}_{tip} - \mathbf{R}_T\left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_T\right) + \mathbf{R}_T\vec{\varepsilon}_T + \vec{\mathbf{p}}_T\right) \\
&= \Delta\mathbf{F}_S^{-1}\left(\mathbf{R}_S^{-1}\mathbf{R}_T\vec{\mathbf{p}}_{tip} - \mathbf{R}_S^{-1}\mathbf{R}_T\left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_T\right) + \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\varepsilon}_T + \mathbf{R}_S^{-1}\left(\vec{\mathbf{p}}_T - \vec{\mathbf{p}}_S\right)\right) \\
&\approx \begin{pmatrix} \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\mathbf{p}}_{tip} - \mathbf{R}_S^{-1}\mathbf{R}_T\left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_T\right) + \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\varepsilon}_T + \mathbf{R}_S^{-1}\left(\vec{\mathbf{p}}_T - \vec{\mathbf{p}}_S\right) \\ + \left(\mathbf{R}_S^{-1}\mathbf{R}_T\vec{\mathbf{p}}_{tip} - \mathbf{R}_S^{-1}\mathbf{R}_T\left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_T\right) + \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\varepsilon}_T + \mathbf{R}_S^{-1}\left(\vec{\mathbf{p}}_T - \vec{\mathbf{p}}_S\right)\right) \times \vec{\alpha}_S \\ - \vec{\varepsilon}_S \end{pmatrix} \\
&= \begin{pmatrix} \vec{\mathbf{p}}_{St} - \mathbf{R}_S^{-1}\mathbf{R}_T\left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_T\right) + \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\varepsilon}_T + \vec{\mathbf{p}}_{St} \times \vec{\alpha}_S \\ + \left(-\mathbf{R}_S^{-1}\mathbf{R}_T\left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_T\right) + \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\varepsilon}_T\right) \times \vec{\alpha}_S \\ - \vec{\varepsilon}_S \end{pmatrix} \\
&\approx \vec{\mathbf{p}}_{St} - \mathbf{R}_S^{-1}\mathbf{R}_T\left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_T\right) + \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\varepsilon}_T + \vec{\mathbf{p}}_{St} \times \vec{\alpha}_S - \vec{\varepsilon}_S
\end{aligned}$$

So, $\vec{\varepsilon}_{St} \approx -\mathbf{R}_S^{-1}\mathbf{R}_T\left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_T\right) + \mathbf{R}_S^{-1}\mathbf{R}_T\vec{\varepsilon}_T + \vec{\mathbf{p}}_{St} \times \vec{\alpha}_S - \vec{\varepsilon}_S$

C. Give an "exact expression" (in terms of the $\Delta\mathbf{R}$'s, etc.) for the error in the value of $\vec{\mathbf{a}}_{tip}$.

$$\begin{aligned}
\vec{\mathbf{a}}_{tip} &= \mathbf{R}_R\vec{\mathbf{p}}_{St} + \vec{\mathbf{p}}_R \\
\vec{\mathbf{a}}_{tip}^{*} &= \mathbf{R}_R\Delta\mathbf{R}_R\left(\vec{\mathbf{p}}_{St} + \vec{\varepsilon}_{St}\right) + \vec{\mathbf{p}}_R + \mathbf{R}_R\vec{\varepsilon}_R \\
\vec{\varepsilon}_{tip} &= \mathbf{R}_R\Delta\mathbf{R}_R\left(\vec{\mathbf{p}}_{St} + \vec{\varepsilon}_{St}\right) + \vec{\mathbf{p}}_R + \mathbf{R}_R\vec{\varepsilon}_R - \left(\mathbf{R}_R\vec{\mathbf{p}}_{St} + \vec{\mathbf{p}}_R\right) \\
&= \mathbf{R}_R\Delta\mathbf{R}_R\left(\vec{\mathbf{p}}_{St} + \vec{\varepsilon}_{St}\right) + \mathbf{R}_R\vec{\varepsilon}_R - \mathbf{R}_R\vec{\mathbf{p}}_{St}
\end{aligned}$$

Where $\vec{\varepsilon}_{St}$ is as computed in part A.

D. Give the corresponding linearized expression in terms of the $\vec{\alpha}_k$'s.

$$\begin{aligned}
\vec{\mathbf{a}}_{tip} &= \mathbf{R}_R\vec{\mathbf{p}}_{St} + \vec{\mathbf{p}}_R \\
\vec{\mathbf{a}}_{tip}^{*} &\approx \mathbf{R}_R\left(\vec{\mathbf{p}}_{St} + \vec{\varepsilon}_{St} + \vec{\alpha}_R \times \left(\vec{\mathbf{p}}_{St} + \vec{\varepsilon}_{St}\right)\right) + \mathbf{R}_R\vec{\varepsilon}_R + \vec{\mathbf{p}}_R \\
&\approx \vec{\mathbf{a}}_{tip} + \mathbf{R}_R\left(\vec{\alpha}_R \times \vec{\mathbf{p}}_{St} + \vec{\varepsilon}_{St} + \vec{\varepsilon}_R\right) \\
\vec{\varepsilon}_{tip} &= \mathbf{R}_R\left(\vec{\alpha}_R \times \vec{\mathbf{p}}_{St} + \vec{\varepsilon}_{St} + \vec{\varepsilon}_R\right)
\end{aligned}$$

Question 4: Constraint propagation & analysis

Suppose that we know that the components of tracker error are limited by constraints

$$\left|\vec{\alpha}_{T,i}\right| \le \mu \qquad \left|\vec{\alpha}_{S,i}\right| \le \mu$$

$$\left|\vec{\varepsilon}_{T,i}\right| \le \nu \qquad \left|\vec{\varepsilon}_{S,i}\right| \le \nu$$

and that the pointer is perfectly calibrated so that $\vec{\varepsilon}_{tip} = \vec{0}$.

A. Give a linearized set of constraints restricting the possible error in the registration transformation $\mathbf{F}_R$, as computed by the process in Question 2. I.e., develop a set of linear inequalities that must be obeyed by $\vec{\alpha}_R$ and $\vec{\varepsilon}_R$.

Let $\vec{\mathbf{c}}_k$ be the computed position $\vec{\mathbf{p}}_{St}$ of the pointer tip when point $\vec{\mathbf{a}}_k$ is measured and $\vec{\mathbf{c}}_k^*$ be the corresponding actual position. Then we know that

$$
\begin{aligned}
\vec{\mathbf{a}}_k &= \mathbf{F}^*{}_R \vec{\mathbf{c}}_k^* \\
&= \mathbf{R}_R \Delta \mathbf{R}_R \left(\vec{\mathbf{c}}_k + \Delta \vec{\mathbf{c}}_k\right) + \vec{\mathbf{p}}_R + \mathbf{R}_R \vec{\varepsilon}_R \\
&\approx \vec{\mathbf{c}}_k + \mathbf{R}_R \left(\vec{\alpha}_R \times \vec{\mathbf{c}}_k + \Delta \vec{\mathbf{c}}_k + \vec{\varepsilon}_R\right) \\
&= \vec{\mathbf{c}}_k + \mathbf{R}_R \vec{\varepsilon}_K + \mathbf{R}_R \vec{\varepsilon}_R - \mathbf{R}_R \left(\vec{\mathbf{c}}_k \times \vec{\alpha}_R\right) \\
&= \vec{\mathbf{c}}_k + \mathbf{R}_R \vec{\varepsilon}_K + \mathbf{R}_R \vec{\varepsilon}_R - \mathbf{R}_R \mathit{skew}(\vec{\mathbf{c}}_k) \vec{\alpha}_R \\
\mathbf{R}_R{}^T \left(\vec{\mathbf{a}}_k - \vec{\mathbf{c}}_k\right) &= \vec{\varepsilon}_k + \vec{\varepsilon}_R - \mathit{skew}(\vec{\mathbf{c}}_k) \vec{\alpha}_R
\end{aligned}
$$

Now, all we need to do is an expression for $\vec{\varepsilon}_k$ in terms of the $\vec{\alpha}_{T,k}$, etc. Dropping the *k* subscript for a moment. We note that this is just the answer to Question 3.B. So, our final constraint set is

$$
\begin{aligned}
\vec{\varepsilon}_k &= -\mathbf{R}_S{}^{-1} \mathbf{R}_T \left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_{T,k}\right) + \mathbf{R}_S{}^{-1} \mathbf{R}_T \vec{\varepsilon}_{T,k} + \vec{\mathbf{c}}_k \times \vec{\alpha}_{S,k} - \vec{\varepsilon}_{S,k} \\
\mathbf{R}_R{}^T \left(\vec{\mathbf{a}}_k - \vec{\mathbf{c}}_k\right) &= \vec{\varepsilon}_k + \vec{\varepsilon}_R - \mathit{skew}(\vec{\mathbf{c}}_k) \vec{\alpha}_R \\
0 &\le \left(\left\|\vec{\alpha}_{S,k}\right\|\right], \left\|\vec{\alpha}_{T,k}\right\| \le \mu \\
0 &\le \left\|\vec{\varepsilon}_{S,k}\right\|, \left\|\vec{\varepsilon}_{T,k}\right\| \le \nu
\end{aligned}
$$

B. Assume now that $\mu = 0.01$ and $\nu = 0.2$ mm and that

$\vec{\mathbf{b}}_1 = [100,100,100]$

$\vec{\mathbf{b}}_2 = [110,100,100]$

$\vec{\mathbf{b}}_3 = [100,110,100]$

$\vec{\mathbf{p}}_{tip} = [0,0,-100]$

$\mathbf{F}_S = [\mathbf{I},[0,-50,-1000]]$

$\mathbf{F}_{T,1} = [\mathbf{I},[0,0,-1000]] = \mathbf{F}_T$ when probe tip touches landmark 1

$\mathbf{F}_{T,2} = [\mathbf{I},[10,0,-1000]] = \mathbf{F}_T$ when probe tip touches landmark 2

$\mathbf{F}_{T,3} = [\mathbf{I},[0,10,-1000]] = \mathbf{F}_T$ when probe tip touches landmark 3

Roughly, how accurate will the computed value of $\vec{\mathbf{a}}_{tip}$ be?

Note: There was a typo here, I had meant to say $\mu = 0.001$. The TA's will be reasonable in assigning credit.

$$
\begin{aligned}
\vec{\varepsilon}_k &= -\mathbf{R}_S{}^{-1} \mathbf{R}_T \left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_{T,k}\right) + \mathbf{R}_S{}^{-1} \mathbf{R}_T \vec{\varepsilon}_{T,k} + \vec{\mathbf{c}}_k \times \vec{\alpha}_{S,k} - \vec{\varepsilon}_{S,k} \\
&= -\left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_{T,k}\right) + \vec{\varepsilon}_{T,k} + \vec{\mathbf{c}}_k \times \vec{\alpha}_{S,k} - \vec{\varepsilon}_{S,k} \\
&= \vec{\varepsilon}_{ptr} + \vec{\varepsilon}_{ref} \\
\vec{\varepsilon}_{ptr} &= -\left(\vec{\mathbf{p}}_{tip} \times \vec{\alpha}_{T,k}\right) + \vec{\varepsilon}_{T,k} \\
\vec{\varepsilon}_{ref} &= \vec{\mathbf{c}}_k \times \vec{\alpha}_{S,k} - \vec{\varepsilon}_{S,k}
\end{aligned}
$$

The value of the three $\vec{\mathbf{c}}_k$ will be

$$
\begin{aligned}
\vec{\mathbf{c}}_1 &= [0,-50,0] \\
\vec{\mathbf{c}}_2 &= [10,-50,0] \\
\vec{\mathbf{c}}_3 &= [10,-40,0]
\end{aligned}
$$

This gives

$$
\begin{aligned}
\left\|\vec{\varepsilon}_{ptr}\right\|_{\max} &\approx 100\mu + \nu = 1.2 \text{ mm} \\
\left\|\vec{\varepsilon}_{ref}\right\|_{\max} &\approx 51\mu + \nu = 0.7 \text{ mm} \\
\left\|\vec{\varepsilon}_k\right\|_{\max} &\approx 1.9 \text{ mm}
\end{aligned}
$$

Now, the big problem is that the sample points are very close together, which will produce a rather large limit on $\vec{\alpha}_R$. Our expression for the error in tip is

$$\vec{\varepsilon}_{tip} \approx \mathbf{R}_R\left(\vec{\alpha}_R \times \vec{\mathbf{p}}_{St} + \vec{\varepsilon}_{St} + \vec{\varepsilon}_R\right)$$
$$= \vec{\alpha}_R \times \vec{\mathbf{p}}_{St} + \vec{\varepsilon}_{St} + \vec{\varepsilon}_R \quad \text{for } \mathbf{R}_R = \mathbf{I}$$

We can estimate limits on

$$\|\vec{\alpha}_R\| \le \frac{2\times 1.9 \text{ mm}}{10 \text{ mm}} = .38$$
$$\|\vec{\varepsilon}_R\| \le 1.9 \text{ mm}$$

For a typical value of $\vec{\mathbf{p}}_{St} = [0,50,0]$ this gives

$$\|\vec{\varepsilon}_{tip}\| \approx .38\times 50 + 1.9 + 1.9 = 22.8 \text{ mm}$$

This tells us that this registration is likely to be fundamentally unusable. To improve it, one either needs a more accurate tracker or a larger baseline on the registration points, or both. E.g., if the value of μ were 0.001, then we would have

$$\|\vec{\varepsilon}_k\|_{max} \approx 151\mu + 2\nu \approx .15 + .4 \approx .6 \text{ mm}$$

If the spacing between the registration landmarks were then made about 40 mm, the resulting value for $\vec{\alpha}_R$ and $\vec{\varepsilon}_R$ would be

$$\|\vec{\alpha}_R\| \le \frac{2\times .6 \text{ mm}}{40 \text{ mm}} = 0.03$$
$$\|\vec{\varepsilon}_R\| \le .6 \text{ mm}$$

For a typical value of $\vec{\mathbf{p}}_{St} = [0,50,0]$ this gives

$$\|\vec{\varepsilon}_{tip}\| \approx .03\times 50 + .6 + .6 = 2.7 \approx 3 \text{ mm}$$

which is more reasonable, though not terrific.

Question 5: Vibration & Disturbance analysis

Suppose that the tracking system, which we have assumed to be stationary in the OR is jostled so that its orientation is changed by an amount

$$\Delta\mathbf{F}_{cam} \approx [\mathbf{I} + skew(\vec{\alpha}_{cam}), \vec{\varepsilon}_{cam}]$$

where

$$\vec{\alpha}_{cam} = [0.01, 0.02, 0.03]$$
$$\vec{\varepsilon}_{cam} = [1,2,3]$$

Assume also that we have the nominal measurements described in Question 4B.

A. What error will be introduced in the measured position of the pointer tip relative to the camera of landmark 1?

$$\vec{\varepsilon}_{added} \approx \vec{\alpha}_{cam} \times \left(\vec{\mathbf{p}}_T + \vec{\mathbf{p}}_{tip}\right) + \vec{\varepsilon}_{cam}$$
$$= [0.01, 0.02, 0.03] \times [0,0,-1100] + [1,2,3]$$
$$= [-21,-11,0] + [1,2,3] = [-20,-9,3]$$

B. What error will be introduced into the computed value of $\vec{\mathbf{a}}_{tip}$?

To a first approximation, no significant error will be introduced.

Question 6 (Extra Credit)

Under the assumptions of Question 4B, how much should the radius of the "safety tube" be reduced to ensure that the actual pointer never goes out of the safe region.

Note: There was a typo here, I had meant to say $\mu = 0.001$ in Question 4B. The TA's will be reasonable in assigning credit.

Essentially, we need to consider the tip and shaft direction uncertainty. Let the uncertainty of the tip position be limited by $\|\vec{\varepsilon}_{tip}\| \le \zeta$ and the shaft direction uncertainty be limited by an angle β. Then the worst case will occur when the tip is offset at the bottom of the pedicle safe tube by ζ and the shaft is tilted in the same direction by an angle uncertainty β. Now, is the length of the safe tube is d_{12}, then the worst case error will be $\zeta + \beta d_{12}$. Plugging these numbers into the assumptions (after fixing the typo) will give $(2.7 + 0.03\, d_{12})$. For a plausible value of d_{12} of 10 mm, this gives 3 mm.